Do Composite Objects Have an Age in Relativistic Spacetime?

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1. Introduction

Many material objects come to be and cease to exist. It is customary to speak of their age. In the classical spacetime framework, the age of an object can be used to label its momentary locations – three-dimensional slices of a four-dimensional path in spacetime. This comes in handy in some metaphysical discussions, such as the debate about persistence.[1]

The situation becomes more complex in the framework of relativity. In Minkowski spacetime, momentary locations of *non-extended* point-like objects can certainly be tracked, labeled or indexed with their *proper time* – the invariant time τ measured along their trajectories:

$$\tau = \int \frac{dt}{\gamma} = \int \sqrt{1 - \frac{v(t)^2}{c^2}}\,dt \tag{1a}$$

$$\tau = \int \sqrt{\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right)}\,ds \tag{1b}$$

$$\tau = \int_L \sqrt{dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2}} \tag{1c}$$

(1a) and (1b) are calculated in a given Cartesian coordinate system (t, x, y, z), and s is a real-valued parameter that can be used to define a spacetime trajectory or path of a material point: $t = t(s)$, $x = x(s)$, $y = y(s)$, and $z = z(s)$. Alternatively, τ can be calculated in terms of a line integral along the object's path L, as in (1c).

[1] For details, see Balashov 2010: Ch. 4.

There is a good sense in which τ can represent the age of such an object (if the object has a finite age).

But what about composite objects consisting, say, of many particles in complex relative motion? Is there a well-defined notion of age for them? And for that matter, is there a well-defined notion of proper time for them? Even if we restrict instantaneous locations of such objects to flat spacelike hypersurfaces[2] they will, in general, "crisscross," even within the object's path, and it is not immediately obvious how one is supposed to identify, label or order them. More precise outlines of the problem will emerge shortly. Here I hasten to note that although in many situations one can simply abstract from the size and composite nature of material objects and continue to work with point idealizations, sooner or later the issue needs to be discussed. And there may be independent interest, both physical and philosophical, in raising it. It is interesting to know whether the notion of age can be coherently applied to composite objects in Minkowski or general relativistic spacetime, and if so, whether there is a good procedure for its determination. Surprisingly, the issue has rarely been discussed. Below I attempt to remedy that situation and offer some comments.

2. Tangential Worries: Metaphysics of Composition

Do composite materials objects have an age? Raising this question may bring with it some interesting and famous, but tangential problems having to do with the metaphysics of composition, which I would like to set aside here. In this particular case, the worry boils down to the question of when a given composite object comes into existence. What defines the beginning of its career and a zero point from which we could start tracking its age? Suppose we have *n* sufficiently scattered particles that come together to compose object *o*. When exactly does it happen? And how can we be sure that *o* maintains its existence later on? Important as these questions may be there is nothing particularly relativistic about them, and they are logically independent of the issues I wish to discuss here.

Accordingly, I will simply assume that these more metaphysical concerns can be put to rest and we can focus on other important questions. In fact, the underlying situation I would like to presuppose is a situation in which a certain composite object starts its career at a certain moment of time t_0 in a certain frame of reference (Figure 1) and never goes out of existence. The particles composing it pursue their separate trajectories in Minkowski spacetime. The interesting question then is: how can we track the career of the whole object and measure its age?

[2] That is, hyperplanes of simultaneity. See Balashov 2010: §5.2 for an argument in favor of such a restriction.

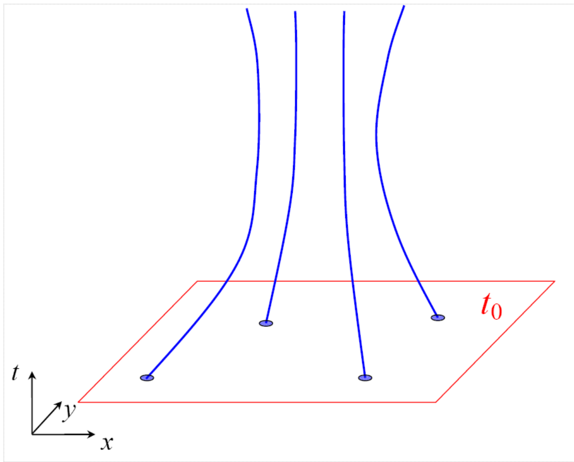


Figure 1. A composite object comes into existence at t_0 in (t, x, y, z).

3. It's Not Easy!

Initially one might think that the task should be relatively easy. After all, we have all these particles and their proper times (Eqs. 1a–1c); so one might hope that, somehow or other, they would "average out." Perhaps we can take an initial clue from a classical case, where it is natural and trivial to associate the spacetime trajectory of a composite object *o* with the trajectory of its *center of mass* (the bold line in Figure 2):

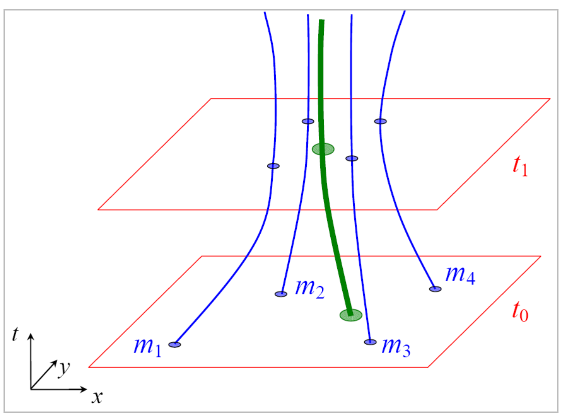


Figure 2. Spacetime trajectory of the center of mass of a composite object in classical spacetime.

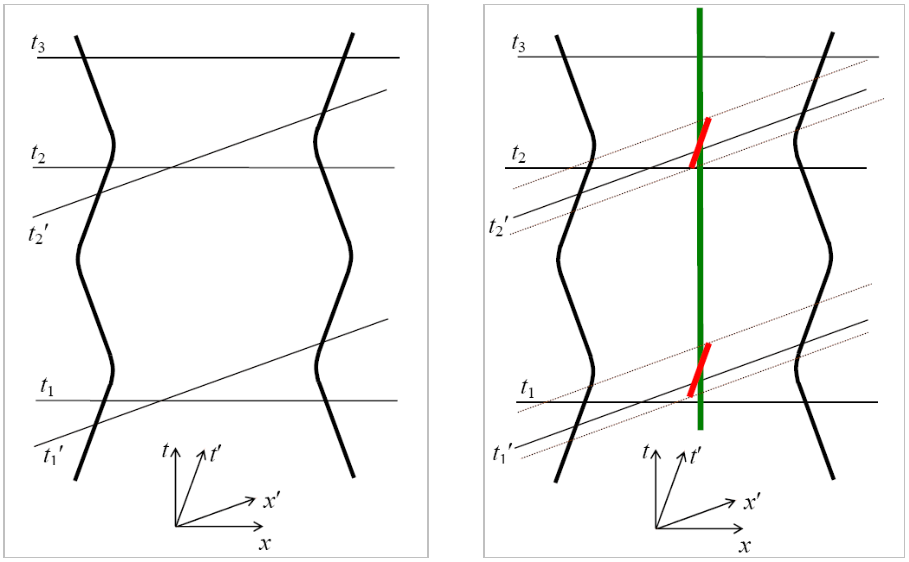
where the radius vector of the center of mass $\mathbf{r}_o$ at any given moment of time is simply the weighted sum of the radius vectors of the components:

$$\mathbf{r}_o = \Sigma m_i \mathbf{r}_i / \Sigma m_i \tag{2}$$

But any attempt to extrapolate this formula to the relativistic context immediately raises a host of questions. Should the masses in question be rest masses or relativistic masses? And if relativistic then in what frame should they be calculated? Relatedly, (2) involves 3-vectors and refers to a particular moment of time. But in what frame? Presumably, in the *instantaneous rest frame* of the whole object. But in order to know in which frame the object is "instantaneously at rest" in the case of n constituent particles in a complicated state of relative motion it would appear that we already need to know what trajectory in spacetime represents the motion of the "object as a whole," and it is unclear that this could be known without knowing the trajectory of the object's center of mass. We seem to be in a circle. In addition, we cannot simply assume, as we do in classical mechanics, that the frame in which the object as a whole is at rest must automatically coincide with the frame in which the total momentum is zero. We can *decide* that this should be the case. Natural though it may seem, it would be a substantive decision.

One still hopes that there should be a reasonably straightforward way out of this circular mess. This hope, however, is dashed rather dramatically by considering a case of an object (Gibson and Pooley 2006: 194, note 29) composed of two oscillating point particles of equal mass, moving uniformly towards and away from each other at the same speed (Figure 3a) in frame (x,t). Obviously the object as a whole is at rest at any moment in this frame: at t_1, t_2, t_3, etc. But it is *also* periodically at rest in a different frame (x',t') co-moving with one of the particles: e.g., at t_1' and t_2'. So the object is at rest in *both* frames that are in relative motion!

This shows that the instantaneous rest frame of a composite object is not an easily-defined concept. Note that this is shown *independently* of evaluating the prospects of any candidate for the role of the center of mass. And when it comes to the latter, the symmetry line of the diagram (Figure 3b) is an obvious candidate for the trajectory of the center of mass of the composite object. But a line that would include the oblique fragments plus some fragments of the symmetry line would also be a good candidate.



(a) (b)

Figure 3. A composite object is at rest in two different frames of reference.

Another curious, even if less realistic, case[3] includes an object composed of a linear array of *infinitely* many identical point particles, each receding from its neighbor at the same relative velocity v. The spacetime trajectory of *any* such particle – or, for that matter, of any symmetry line of this configuration – could be taken to represent, equally well, the trajectory of the whole object – an extreme case in point (Figure 4)! Below I abstract from such examples involving an infinite number of material parts and focus on a system of n particles.

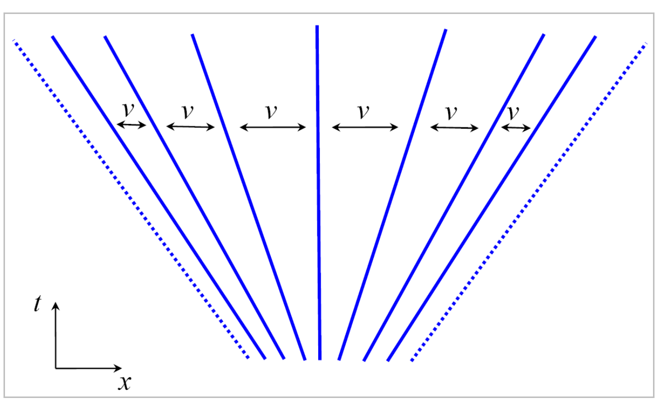


Figure 4. An object composed of an infinite number of mutually receding particles is at rest in an infinite number of reference frames.

[3] Suggested by Cody Gilmore (personal communication).

Is there *any* general way to define a unique trajectory representing, somehow or other, the motion of an arbitrary composite object in Minkowski spacetime? To sum up the problem so far, in order to determine the trajectory of the center of mass we need to calculate all the quantities in formula (2) above at a moment of time in the instantaneous rest frame of the whole object. But in order to know which frame is the instantaneous rest frame we need to know the trajectory of the center of mass. Cases such as those in Figures 3 and 4 strongly suggest that there is no easy way out of this circular mess.

4. A Non-starter: Synchronize the Clocks

Before moving on I would like to consider and set aside another proposal to which one might be led by a desperate desire to avoid dealing with the circular mess. This proposal is similar to one considered and rejected by Gilmore (2008: 1239–1240) in a different context. The idea is to attach a small clock to each particle, set them all to zero at t_0, then track the proper time of each particle with its corresponding clock, and then simply mark the locations of all the particles after 1 second, after 2 seconds, etc. of their proper times. Once we have these locations we can draw hypersurfaces through them and identify the resulting filled regions with the locations of the whole object at the age of 1 second, at the age of 2 seconds, etc. (Figure 5). And once we have such locations we can, if we wish, determine the position of the center of mass at each of them and then connect them, thereby producing a spacetime trajectory of the whole object.

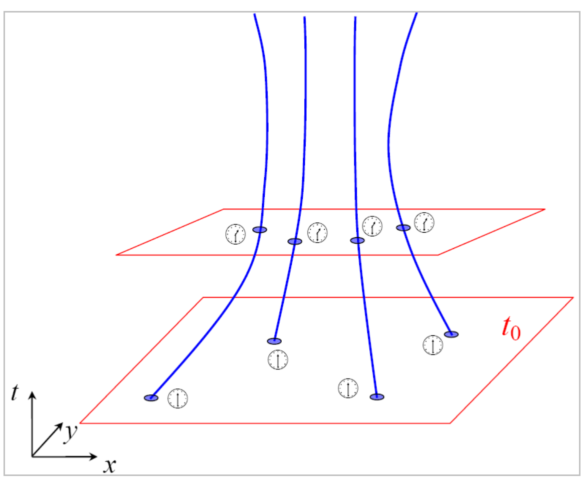


Figure 5. Synchronize the clocks!

This proposal is untenable because the resulting regions defined according to its prescription will quickly go wild. At some point they will stop being spacelike and even sooner they will stop being flat. This is easy to see if we help ourselves to a small "twins scenario." Call one particle Alice and another Bob (Figure 6). Alice comes back to reunite with Bob, and continues to stay with him, and she is so much younger. So if we wanted to synchronize their ages in the way suggested we would need to put the 20-year old Alice at a point timelike separated from the location of the 20-year old Bob. When considered at these two locations, Alice and Bob cannot compose anything worthwhile.

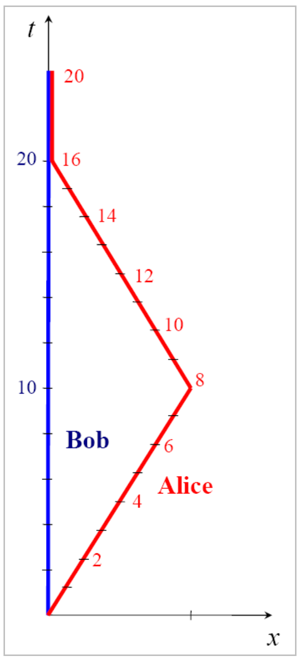


Figure 6. Alice and Bob.

5. Back to the Circular Mess: The Procedure

So we do need to deal with the circular mess. Is there *any* general way to define a trajectory representing, somehow or other, the motion of an arbitrary composite object in Minkowski spacetime? One would expect there to be some history of the discussion of this question and some authoritative work. And there is; but it is scanty.[4] Pryce (1948) and Schattner (1978, 1979), in particular, are frequently cited in later developments.[5] The interest in the problem seems to have been driven by rather diverse motivations ranging from predominantly mathematical curiosity to attempts to use the resulting constructions as a bridge between the micro and the macro to draw some rough-and-ready consequences for the foundations of quantum physics.[6] The exact details of the more serious developments lie beyond my mathematical expertise. But I wish to note a convergence of some of these developments with my own ideas, to which I was led before becoming aware of this larger literature (Balashov 2010: 191–195). Accordingly, I will take the liberty to sketch my toy procedure to determine the worldline of an arbitrary object composed of *n* non-interacting particles, in Minkowski spacetime. It is far from rigorous and has other limitations too. But it will allow me to illustrate the basic idea in simple terms. I discuss the limitations and necessary refinements in section 6.

The basic idea, in the idealized case of *n* non-interacting particles, is to chart the trajectory of a composite object by connecting the locations of its center of mass determined in instantaneous frames in which the total momentum is zero, using relativistic quantities (i.e. dynamic masses, etc.), and then translate the result to an arbitrary frame by a Lorentz transformation. This is then how the circle could be broken – by identifying the zero-momentum frame first.

In a bit more detail:[7] consider object o composed of n particles o_1, o_2, … o_n with continuous and smooth trajectories $\mathbf{r}_i = \mathbf{r}_i(s)$, $t = t(s)$ in a coordinate system

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[4] See, in particular, Fokker 1929, Papapetrou 1940, Pryce 1948, Møller 1949, Madore 1969, Dyxon 1970ab, Ehlers and Rudolph 1977, Schattner 1978, 1979, Bailey and Israel 1980, Chryssomalakos et al. 2009, Mermin 2011.

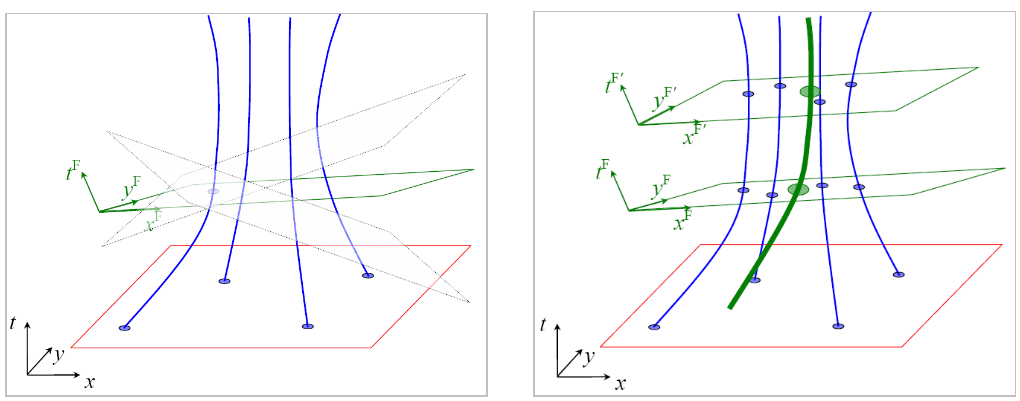
[5] My thanks to Oliver Pooley for drawing my attention to these important works.

[6] For the latter, see Chryssomalakos et al. 2009. A curious recent development is a short note by N. David Mermin (2011) responding to H. C. Ohanian's claim that Einstein made several mistakes in his famous 1905 derivation of the mass-energy formula. One of these mistakes, according to Ohanian, includes failure to define the velocity of a composite body, as "there is no obvious 'fiducial point', such as the nonrelativistic center of mass, whose velocity can be used to represent the velocity of the body as a whole" (Mermin 2011: 1). Mermin responds by noting that "if the body is indeed a body – if the internal motions of its parts do not take them more than a bounded distance away from one another – then it is clear how to identify the rest frame to any desired degree of precision. The rest frame is that unique frame in which, no matter how long you wait, part of the body can be found within some bounded region that originally contained the entire body" (ibid.). Mermin concludes, "So there is no problem in defining the velocity of an extended body, even when its parts are in relative motion, and even if their relative velocities are comparable to the speed of light *c*" (ibid.). While this proposal may address a particular issue raised by Ohanian in the context of Einstein's derivation it can hardly serve as a general recipe for defining a unique trajectory representing the motion of an arbitrary composite object in relativistic spacetime. Witness the simple two-body case considered above (Figure 3). My thanks to Geurt Sengers for drawing my attention to Mermin's note.

[7] The outline of the toy procedure below follows Balashov 2010: 191–195.

$(\mathbf{r},t)$ adapted to some inertial reference frame, where s is a real-valued parameter. We are looking for a trajectory $\mathbf{r}_o = \mathbf{r}_o(s)$, $t_o = t_o(s)$ representing (somehow or other) the motion of o. Choose some particle o_1 and its location $(\mathbf{r}_1(s),t(s))$, for some value of s. The most important step then is to identify a time hyperplane through $(\mathbf{r}_1(s),t(s))$, at which the total 3-momentum of o is zero. That is to say, we should identify a reference frame F(s) (an "instantaneous rest frame of o") such that, for some coordinate system $(\mathbf{r}^{\mathrm{F}},t^{\mathrm{F}})$ adapted to F, a particular time hyperplane $t^{\mathrm{F}}=t^{\mathrm{F}}(s)$ contains $(\mathbf{r}_1(s),t(s))$ and $|\Sigma m_i^{\mathrm{F}}\mathbf{v}_i^{\mathrm{F}}|=0$, where all the $m_i^{\mathrm{F}}\mathbf{v}_i^{\mathrm{F}}$'s are calculated at $t^{\mathrm{F}}=t^{\mathrm{F}}(s)$ in $(\mathbf{r}^{\mathrm{F}},t^{\mathrm{F}})$.

Less formally: draw various time hyperplanes through $(\mathbf{r}_1(s),t(s))$ and find one (the solid hyperplane in Figure 7a) that yields zero total momentum. There is every reason to call the associated frame of reference an *instantaneous rest frame* of the whole object. Then find the radius vector of the center of mass $C^{\mathrm{F}}(s)$ of o at $t^{\mathrm{F}}=t^{\mathrm{F}}(s)$ in $(\mathbf{r}^{\mathrm{F}},t^{\mathrm{F}})$: $\mathbf{r}^{\mathrm{F}}{}_o = \Sigma m_i^{\mathrm{F}}\mathbf{r}_i^{\mathrm{F}}/\Sigma m_i^{\mathrm{F}}$. Now repeat the whole procedure for other values of s. Connect the locations of $C^{\mathrm{F}}(s)$ thus obtained (Figure 7b). Finally, transform the positions $(\mathbf{r}^{\mathrm{F}}{}_o(s),t^{\mathrm{F}}{}_o(s))$ of all the $C^{\mathrm{F}}(s)$'s to the original coordinate system $(\mathbf{r},t)$.



(a) (b)

Figure 7. Toy procedure for drawing the worldline of an arbitrary object composed of n non-interacting particles in Minkowski spacetime.

6. Limitations of the Toy Procedure and Rigorous Developments

The toy procedure sketched above is rather convoluted, some steps in any given cycle in it are implicit, and different cycles are not coordinated with each other. Will the procedure generate a unique, continuous and smooth trajectory? The question cannot be answered without undertaking a more rigorous approach.

Some limitations of the toy procedure have to do with neglecting interaction among o's constituent particles. In the absence of interaction, the notion of the common center of mass of o_1, o_2, … o_n seems to be a somewhat arbitrary quantity without well-defined physical meaning.[8] One way to add some "thickness" to the notion is to associate it with a particular dynamical role perhaps similar to the role of the center of mass in classical mechanics where it is, essentially, the *center of balance*. However, in relativistic mechanics stresses in media are connected with energy densities in unusual ways and themselves contribute to the dynamic mass of the system. Accordingly, there is no way around starting with the stress energy tensor.

An approach (whose details outstrips my expertise) along these lines was developed in a more technical environment by Pryce (1948), and his method was then extended to general relativity by Madore (1969), Dyxon (1970ab), Ehlers and Rudolph (1977), and Schattner (1978, 1979).[9] Pryce begins by considering *six* different methods for defining the center of mass of a system of free particles in special relativity, most of them unsatisfactory, and develops in detail one promising strategy, which he also traces back to Fokker (1929). Pryce then expresses the solution in a form that allows him to extrapolate it to the case of interacting particles. This is done in two steps by starting with a kinematic expression for the four-momentum of a system of *n* free particles in a given frame:[10]

$$P^{\mu} \equiv \sum_{i=1}^{n} p_i^{\mu} \qquad (3)$$

and the following (preliminary) method, whereby the coordinates of the center of mass of the whole system q^{μ} are identified with the mean of the coordinates of the constituent particles q_i^{μ} weighted with their *relativistic* mass-energies in *that* frame:

$$P^0 q^{\mu} = \sum_i p_i^0 q_i^{\mu} \qquad (4)$$

(4) can be usefully viewed as an analog of (2). Pryce then re-expresses $P^0 q^{\mu}$ in terms of the energy-momentum tensor of a system of free particles

[8] Perhaps somewhat similar to the notion of the "center of population" of a country. I thank John Norton for the analogy.

[9] I would like to think that despite its obvious limitations the toy model sketched above is in line with these systematic developments. But I will leave it to others to see if the similarity is close enough to use the toy model as a good illustration of the rigorous approach.

[10] The outline below of the rigorous procedure closely follows Pryce 1948: 64–65 with some minor change of notation.

$$T^{\mu\nu}(\mathbf{x}) = \sum_i \int \delta(x^0 - q_i^0)\,\delta(x^1 - q_i^1)\,\delta(x^2 - q_i^2)\,\delta(x^3 - q_i^3)\,p_i^\mu dq_i^\mu \qquad (5)$$

as follows:

$$P^0 q^\mu = \iiint x^\mu T^{00} dx^1 dx^2 dx^3 \,, \qquad (6)$$

which also suggests another tensor quantity $M^{\mu\nu}$ for the role of representing the total angular momentum:

$$M^{\mu\nu} = \iiint (x^\mu T^{0\nu} - x^\nu T^{0\mu}) dx^1 dx^2 dx^3 \,, \qquad (7)$$

This results in a simple expression

$$q^\mu = (tP^\mu + M^{\mu 0}) / P^0 \,, \qquad (8)$$

where $t = x^0$. According to Pryce, (8) "can be applied to a system of particles interacting through a field, thereby removing the original limitation to free particles" (1948: 65).

As it turns out, this simple definition is not independent of the frame of reference. Transforming it to another frame gives rise to extraneous terms. At this point Pryce ties one of the resulting expressions to a frame in which the total momentum vanishes (the zero-momentum frame) and obtains another more complicated expression:

$$q^\mu = \frac{tP^\mu}{P^0} + \frac{M^{\mu\nu} P_\nu}{m^2} + \frac{M^{\mu 0} P^\mu P_\nu}{m^2 P^0} \,, \qquad (9)$$

"which, in spite of its appearance, is relativistically covariant" (ibid.: 65). Here m is the rest mass of the whole system: $m^2 = P^\mu P_\mu$. I believe (9) is a rigorous counterpart of essentially the same "zero-momentum frame" approach informally outlined in the toy procedure described above – but, of course, without the limitations of the latter.

Based on Pryce's work and some related developments confined to special relativity[11], several authors – in particular, Dixon (1970ab) and Schattner (1978) – formulated similar strategies in the context of general relativity. Furthermore,

[11] In particular, Papapetrou 1939 and Møller 1940.

Schattner (1979) claims to have established the existence and uniqueness results for his definition of a center-of-mass line for an extended body.

If these developments are correct, how do they square with the worry about non-uniqueness raised by Gibson and Pooley's two-particle case mentioned above? What should disqualify the oblique boldface fragments in Figure 3b from being fragments of a distinct center of mass trajectory of this composite object, alongside the symmetry line of the configuration? Perhaps the neglect of external forces that are needed to make the system perform this sort of motion. Taking such forces into account will require introducing a field that will contribute to the determination of the trajectory of the center of mass, along the lines of Pryce's proposal, and any realistic way of doing so is likely to rule out the oblique fragments.

7. How Much Does It All Matter?

How much does all of that matter in talking about the age of mid-sized ordinary objects in metaphysical discussions about persistence, say? One could agree that the exact determination of the age of spatially extended persisting objects becomes difficult, if not impossible. This can be done only approximately, with a certain "margin or error." The main factor responsible for the vagueness of an object's age is the relative motion of its constituent particles, whereby the ages of different particles get progressively "out of step" with each other, due to relativistic time dilation (the "twins effect" illustrated in Figure 6). How large is this factor?

This question may not have a straightforward answer. Indeed, the answer will depend on the choice of a relevant level of structure. Could tables and chairs (cats and dogs, human beings) be taken to be composed of molecules? Or of atoms? Assuming the former for human beings, the relevant speed can be associated with molecular motion, with a conservative upper bound set at 1 km/sec. This corresponds to $\gamma = 1.000000000006$ and translates into the cumulative time difference (between the "ages" of two molecules in constant relative motion) of mere 0.01 sec over the period of 50 years. One could perhaps rest assured that this sort of indeterminacy is completely innocuous. But of course, molecules are not metaphysical atoms. One needs to go deeper, to physical atoms and subatomic particles. And at that point the situation quickly gets out of control. First of all, things start moving much faster. And one cannot abstract from interaction anymore; indeed, interaction becomes the main contributing factor. And on top of it, the classical non-quantum description ceases to be valid.

But it is good to take one step at a time.[12]

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[12] This paper is an offshoot of a larger project (Balashov 2010). My thanks to Oliver Pooley, Nick Huggett, and John Norton for their help, and to Oxford University Press for the permission to use some of the material of section 7.9 of Balashov 2010 (pp. 192–195). Versions of the paper were given at the International Workshop on Temporal Existence and Persistence in Spacetime, University-Club Bonn, Germany (February 2011) and the joint Physics/Philosophy Seminar at Idaho State University (Pocatello, Idaho, USA, April 2011). I am grateful to both audiences for very stimulating discussions and to the organizers for their hospitality. Special thanks are due to Cord Friebe, Thomas Müller, and Florian Fischer for their comments on the draft of this paper.

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